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The relaxed three-algebras: their matrix representation and implications for multi M2-brane theory

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Abstract: We argue that one can relax the requirements of the non-associative three-algebras recently used in constructing $D = 3$ $\mathcal{N} = 8$ superconformal field theories, and introduce the notion of "relaxed three-algebras". We present a specific realization of the relaxed three-algebras in terms of classical Lie algebras with a matrix representation, endowed with a non-associative four-bracket structure which is prescribed to replace the three-brackets of the three-algebras. We show that both the $so(4)$-based solutions as well as the cases with non-positive definite metric find a uniform description in our setting. We discuss the implications of our four-bracket representation for the $D = 3$, $\mathcal{N} = 8$ and multi M2-brane theory and show that our setup can shed light on the problem of negative kinetic energy degrees of freedom of the Lorentzian case.

Keywords: AdS-CFT Correspondence, M-Theory.

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Contents

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1. Introduction

Until recently finding an action for the maximally supersymmetric three-dimensional conformal (gauge) field theory had remained elusive [1−4] (e.g. see [5] for a short review). The $D = 3$, $\mathcal{N} = 8$ superconformal field theory (SCFT) is expected to arise from the "low energy" effective action describing many M2-branes on eleven dimensional Minkowski spacetime. Hence its formulation is closely linked with finding the theory describing N eleven-dimensional membranes. Furthermore, via the AdS/CFT correspondence [6], this SCFT is dual to M-theory on $AdS_4 \times S^7$, the background which is obtained from the geometry corresponding to coincident parallel M2-branes in the near-horizon (decoupling) limit [6].

The $D = 3$, $\mathcal{N} = 8$ SCFT action is invariant under the three-dimensional superconformal group $Osp(8|4)$, with bosonic generators belonging to $so(8) \times usp(4) \simeq so(8) \times so(3, 2)$. Moreover, the action for a single M2-brane enjoys invariance under the area preserving diffeomorphisms (APD's) on the 2+1 dimensional world-volume as its local (gauge) symmetry e.g. [7 – 9]. Thus the multi membrane action is expected to have a gauge symmetry which somehow manifests this local gauge invariance. The mathematical (algebraic) structure which encodes three-dimensional APD's is the Nambu three-bracket [9 – 11]. Therefore, finding an action for the $D = 3$, $\mathcal{N} = 8$ SCFT is ultimately related to quantization of Nambu three-brackets.

It has been argued that although classical Nambu p-brackets ($p \geq 3$) enjoy associativity (e.g. see appendix B of [12]) the "quantized" Nambu p-brackets cannot be associative [8]. For the case of three-brackets, as was proposed originally by Nambu [10], one may use the associator of a non-associative algebra as the quantum version of the three-bracket. In fact this idea was put at work by Bagger and Lambert to construct the action for the $D = 3$, $\mathcal{N} = 8$ SCFT, the BL theory [1, 2], where this non-associative algebra with its three element structure (the associator) was called the *three-algebra*. A *three-algebra no-go theorem* was argued for in [14] and then proved in [15]. This no-go theorem states that the only three-algebra which has a positive definite norm is either $so(4)$ or direct sums of a number of $so(4)$'s. In order to describe N M2-brane theory (for a generic N), similarly to N Dp-brane cases, one would like to be able to write the BL theory with more general algebras whose rank (or dimension) are related to the number of M2-branes and hence bypass this no-go theorem. This theorem can, however, be circumvented by considering algebras of non-positive norm [16 – 18].

In this paper we use another prescription for quantizing the Nambu three-bracket. This prescription was used in [12] to quantize type IIB D3-branes to obtain a matrix theory description for the DLCQ of IIB string theory on the $AdS_5 \times S^5$ or the plane-wave. In this approach we replace the classical Nambu three-brackets with the "quantum" Nambu four-brackets which involve usual matrices. Although the structure of the quantized Nambu four-bracket we obtain is non-associative [12] the underlying algebra, which is nothing but the usual matrix multiplication algebra, is associative. In particular we use $2N \times 2N$ matrices to describe the $D = 3$, $\mathcal{N} = 8$ SCFT corresponding to the low energy limit of N M2-branes.

Our prescription requires an extension or relaxation of the notion of three algebras giving rise to multi M2-brane theories, which will be called *relaxed three-algebras*. Recently modifications on the mathematical conditions defining a three-algebra have been considered. These "generalized" three-algebras are obtained by relaxing the antisymmetry of the three-bracket and metricity of the algebra [13]. Here, instead of focusing on the antisymmetry or metricity of these algebras, we will relax the closure and the fundamental identity conditions in a way to be described below. In our representation for the relaxed three-algebras we show that only the two Euclidean and Lorentzian cases are possible, compatible with results of [15] and [19]. Moreover, we show that for the Lorentzian case the $su(N)$ algebras in $N \times N$ representation are relevant to the theory of N M2-branes. More importantly we show that there is nothing inherently "Lorentzian" in the underlying $su(2N)$ algebra over which the four-bracket structure is defined.

This paper is organized as follows. In section 2, we give a brief review of the BL theory and its supersymmetry and gauge transformations. In section 3, we present the notion of relaxed three-algebras. In section 4, we derive matrix representations for the relaxed three-algebras. This is done through the "four-brackets" which replace the three-brackets of the BL three-algebras. We check these representations satisfy the necessary (relaxed) closure and fundamental identity conditions. In section 5, we discuss the implications of our relaxed three-algebra realizations for the multi M2-brane BL theory. We argue that our prescription, supplemented by arguments of [20, 21], resolves the problem of ghost-type degrees of freedom appearing in the ordinary treatment of the Lorentzian case (see [22] for other ways to resolve the ghost problem). We check that this theory has the necessary properties expected from a $D = 3$, $\mathcal{N} = 8$ SCFT and multi M2-brane action by examining its behaviour under worldvolume parity and spectrum of its 1/2 BPS states. The last section is devoted to concluding remarks and open questions.

2. Review of the BLG theory

In this section to fix the conventions and notations we briefly review the BLG theory by first defining the three-algebras $\mathcal{A}_3$ and their algebraic structure and then presenting the BLG proposed action for the $D = 3$, $\mathcal{N} = 8$ superconformal field theory.

2.1 The BLG three-algebras

The three-algebra $\mathcal{A}_3$ is an algebraic structure defined through the three-bracket $[\![\ ,\ ,\ ]\!]$

$$[\![\Phi_1, \Phi_2, \Phi_3]\!] \in \mathcal{A}_3, \quad \text{for any } \Phi_i \in \mathcal{A}_3, \tag{2.1}$$

where

$$[\![\Phi_1, \Phi_2, \Phi_3]\!] = -[\![\Phi_2, \Phi_1, \Phi_3]\!] = -[\![\Phi_1, \Phi_3, \Phi_2]\!] \tag{2.2}$$

The three-bracket, which is a "quantized" Nambu three-bracket [10] is indeed an associator and $\mathcal{A}_3$ is a *non-associative* algebra. The three-bracket should satisfy an analog of the Jacobi identity, the *fundamental identity* [11]:

$$\begin{aligned}\mathcal{K}_{ijklm} &= [\![[\![\Phi_i, \Phi_j, \Phi_k]\!], \Phi_l, \Phi_m]\!] + [\![[\![\Phi_i, \Phi_j, \Phi_l]\!], \Phi_m, \Phi_k]\!] + [\![[\![\Phi_i, \Phi_j, \Phi_m]\!], \Phi_k, \Phi_l]\!] \\ &= [\![\Phi_i, \Phi_j, [\![\Phi_k, \Phi_l, \Phi_m]\!]]\!].\end{aligned} \tag{2.3}$$

As we can see $\mathcal{K}_{ijklm}$ is anti-symmetric under exchange of the first two as well as the last three indices. We equip this algebra with a product $\bullet$ and a Trace

$$Tr(\Phi_1 \bullet \Phi_2) = Tr(\Phi_2 \bullet \Phi_1) \in \mathbb{C} \tag{2.4}$$

satisfying a "by-part integration" property

$$Tr(\Phi_1 \bullet [\![\Phi_2, \Phi_3, \Phi_4]\!]) = -Tr([\![\Phi_1, \Phi_2, \Phi_3]\!] \bullet \Phi_4). \tag{2.5}$$

For the usage in physical theories, noting that Φ_i's are complex valued, it is needed to define the Hermitian conjugation over the algebra. In particular if we choose to work with Hermitian algebras, i.e.

$$\Phi^\dagger = \Phi, \quad \forall \Phi \in \mathcal{A}_3,$$

then the closure condition (2.1) is satisfied with the following definition for complex conjugation of the three-bracket:

$$[\![\Phi_1, \Phi_2, \Phi_3]\!]^\dagger = [\![\Phi_1^\dagger, \Phi_2^\dagger, \Phi_3^\dagger]\!] \ . \tag{2.6}$$

If we expand $\mathcal{A}_3$ elements in terms of the complete basis T^a

$$\Phi = \Phi_a T^a$$

then (2.1) implies that

$$[\![T^a, T^b, T^c]\!] = f^{abc}{}_d T^d \tag{2.7}$$

and

$$Tr(T^a \bullet T^b) \equiv h^{ab} \tag{2.8}$$

defines the metric h^{ab} on $\mathcal{A}_3$. Mathematically, the metric h^{ab} can have arbitrary signature, though physically, non-positively defined signatures could give rise to ghost degrees of freedom. We will always take h^{ab} to be non-degenerate and invertible. Noting the (2.5),

$$f^{abcd} \equiv f^{abc}{}_e h^{ed},$$

is totally anti-symmetric four-index structure constant. The fundamental identity in terms of the structure constant f is written as

$$f^{abc}{}_l f^{del}{}_m + f^{abd}{}_l f^{ecl}{}_m + f^{abe}{}_l f^{cdl}{}_m = f^{cde}{}_l f^{abl}{}_m. \tag{2.9}$$

This equation does not have any solution other than $f^{abcd} = \epsilon^{abcd}$ or four tensors made out of ϵ^{abcd}, if h^{ab} is positive definite and hence $\mathcal{A}_3$ is either $so(4)$ or combinations involving the direct sums of $so(4)$ [15].

To find three-algebras other than $so(4)$ one is hence forced to relax the positive definite condition on h_{ab} [16, 17]. Explicitly if we choose $a = (+, -, \alpha)$ and

$$h_{\alpha\beta} = \delta_{\alpha\beta},\ h_{+\alpha} = h_{-\alpha} = 0,\ h_{++} = h_{--} = 0,\ h_{+-} = h_{-+} = -1 \tag{2.10}$$

then $f^{abc}{}_d$ with non-zero components

$$f^{\alpha\beta\gamma}{}_- \equiv f^{\alpha\beta\gamma},\ f^{\alpha\beta+}{}_\gamma = f^{\alpha+\beta}{}_\gamma = -f^{\alpha+\beta}{}_\gamma = f^{\alpha\beta\rho}\delta_{\rho\gamma} \tag{2.11}$$

is a solution to the fundamental identity (2.9), provided that $f_{\alpha\beta\gamma}$ are satisfying the usual Jacobi identity for associative algebras [17].

Finally we point out that if T^a's are all Hermitian then with (2.6) the structure constant f_{abcd} should be real valued, that is $f^*_{abcd} = f_{abcd}$.

2.2 The BLG action

The on-shell matter content of the $D = 3$, $\mathcal{N} = 8$ hypermultiplet involves eight three-dimensional scalars X^I, $I = 1, 2, \cdots, 8$ in the 8_v of the SO(8) R-symmetry group, eight two component three-dimensional fermions Ψ in the 8_s of SO(8) (we have suppressed both the $3d$ and the R-symmetry fermionic indices). Each of the above physical fields which will generically be denoted by Φ are also assumed to be elements of the three-algebra and hence

$$\Phi = \Phi_a T^a.$$

The action of the BLG [1–3] theory is given by

$$\begin{aligned} S = \int d^3\sigma\ Tr\Big(-\frac{1}{2} D_i X^I D^i X^I - \frac{1}{2.3!} [\![X^I, X^J, X^K]\!] [\![X^I, X^J, X^K]\!] \\ + \frac{i}{2} \bar{\Psi} \gamma^i D_i \Psi - \frac{i}{4} [\![\bar{\Psi}, X^I, X^J]\!] \Gamma^{IJ} \Psi \Big) + \mathcal{L}_{\text{twisted Chern-Simons}} \end{aligned} \tag{2.12}$$

where $\mathcal{L}_{\text{twisted Chern-Simons}}$ is a parity invariant Chern-Simons action

$$\mathcal{L}_{\text{twisted Chern-Simons}} = \frac{1}{2} \epsilon^{ijk} \left(f^{abcd} A_{i\ ab} \partial_j A_{k\ cd} + \frac{2}{3} f^{abcl} f^{deg}{}_l A_{i\ ab} A_{j\ de} A_{k\ cg} \right). \tag{2.13}$$

Indices $i = 0, 1, 2$ denote the three-dimensional directions and the covariant derivatives are defined as

$$(D_i \Phi)_a \equiv \partial_i \Phi_a - f^{cdb}{}_a\ A_{i\ cd} \Phi_b \tag{2.14}$$

where $A_{i\ ab}$ is the non-propagating three dimensional, two-index gauge field. For later use it is useful to introduce another gauge field

$$\tilde{A}_i{}^b{}_a = f^{cdb}{}_a\ A_{i\ cd}. \tag{2.15}$$

The above action is invariant under the local gauge symmetry:

$$\begin{aligned} \delta_{\text{gauge}} \Phi_a &= f^{cdb}{}_a \Lambda_{cd} \Phi_b, \\ \delta_{\text{gauge}} A_{i\ cd} &= \partial_i \Lambda_{cd} - f^{abe}{}_{[c} \Lambda_{d]e} A_{i\ ab} \end{aligned} \tag{2.16}$$

as well as the global supersymmetry transformations

$$\begin{aligned} \delta_{\text{susy}} X^I &= i\bar{\epsilon} \Gamma^I \Psi \\ \delta_{\text{susy}} \Psi &= D_i X^I \Gamma^I \gamma^i \epsilon - \frac{1}{6} [\![X^I, X^J, X^K]\!] \Gamma^{IJK} \epsilon \\ \delta_{\text{susy}} \tilde{A}_i^{ab} &= i f^{abcd} \epsilon \gamma_i \Gamma_I X_c^I \Psi_d \end{aligned} \tag{2.17}$$

It has also been shown that [14] besides the $2+1$ dimensional super-Poincaré symmetry the above action, at least at classical level, is invariant under the full three-dimensional superconformal algebra.

The equations of motion of the above action are

$$\begin{aligned} \gamma^i D_i \Psi + \frac{1}{2} \Gamma^{IJ} [\![X^I, X^J, \Psi]\!] &= 0 \\ D^2 X^I - \frac{i}{2} \Gamma^{IJ} [\![\bar{\Psi}, X^J, \Psi]\!] + \frac{1}{2} [\![X^J, X^K, [\![X^I, X^J, X^K]\!]]\!] &= 0 \\ \tilde{F}_{ij}^{ab} + \epsilon_{ijk} f^{abcd} \Big(X_c^J D^k X_d^J + \frac{i}{2} \bar{\Psi}_c \gamma^k \Psi_d \Big) &= 0 \end{aligned} \tag{2.18}$$

where

$$\tilde{F}_{ij}{}^{b}{}_{a} = \partial_i \tilde{A}_j{}^{b}{}_{a} - \partial_j \tilde{A}_i{}^{b}{}_{a} - \tilde{A}_i{}^{b}{}_{c}\tilde{A}_j{}^{c}{}_{a} + \tilde{A}_j{}^{b}{}_{c}\tilde{A}_i{}^{c}{}_{a} \ .$$

In the BL theory, for both the Lorentzian and Euclidean realizations of the three-algebras, the basis T^a and hence all the components of the X field X_a are both taken to be Hermitian. It is also worth noting that with this requirement and the Hermiticity property (2.6) the *potential* terms in the Hamiltonian of the BL theory in both Lorentzian and Euclidean cases are positive definite.

3. The relaxed three-algebras

The construction of BLG three-algebras with the definition and properties outlined in section 2.1 has proven very restrictive. In this section we revisit the BL analysis with the idea that we may be able to relax some of the conditions on the BL three-algebras while keeping the physical outcomes intact. We will see this is indeed possible.

As discussed in section 2.1, three-algebras of interest are defined by five conditions: a totally anti-symmetric three-bracket, existence of non-degenerate metric, the closure of the three-algebra under the three-bracket, the fundamental identity and the trace property (2.5). The antisymmetry, closure and fundamental identity are conveniently expressed in terms of a basis T^a and the structure constants $f^{abc}{}_d$ as in (2.7) and (2.9).

Let us *relax* the closure and fundamental identities, while keeping the antisymmetry and the trace property, by enlarging the set of T^a's through the addition of extra generators T^A's satisfying the properties:

i) T^A is orthogonal to every other generator, i.e.

$$Tr(T^aT^A) = 0, \qquad Tr(T^AT^B) = 0 \ . \tag{3.1}$$

ii) T^A in the brackets:

$$[\![T^a, T^b, T^c]\!] = f^{abc}{}_d T^d + k^{abc}{}_A T^A, \tag{3.2a}$$

$$[\![T^a, T^b, T^A]\!] = f^{abA}{}_B T^B, \ [\![T^a, T^A, T^B]\!] = f^{aAB}{}_C T^C, \ [\![T^A, T^B, T^C]\!] = f^{ABC}{}_D T^D, \tag{3.2b}$$

where $f^{abc}{}_d$ are still satisfying the standard fundamental identity (2.9) and any other additional four-index structure constant, i.e. $f^{xyz}{}_A \ \forall\ x,\ y,\ z$, is yet unknown. Notice that the form of $[\![T^a, T^b, T^A]\!]$, $[\![\ T^a, T^A, T^B]\!]$ and $[\![T^A, T^B, T^C]\!]$ is fixed by demanding the consistency of these brackets with the "by-part" property (2.5).

If $k^{abc}{}_A$ are zero, we can just simply ignore the existence of the T^A's and we are back to the BL three-algebra $\mathcal{A}_3$. However, with non-zero $k^{abc}{}_A$, the algebra of T^a's does not close. Nonetheless, we can still have a generalized or *relaxed* notion of closure. If we denote the part of the algebra spanned by T^a's by $\mathcal{K}$ and the part spanned by T^A's by $\mathcal{K}_S$, (3.2) can be rewritten as

$$[\![\Phi_1, \Phi_2, \Phi_3]\!] \in \mathcal{K} \oplus \mathcal{K}_S, \qquad \forall \Phi_i \in \mathcal{K}\,, \tag{3.3a}$$

$$[\![\Phi_1, \Phi_2, \chi]\!],\ [\![\Phi, \chi_1, \chi_2]\!],\ [\![\chi_1, \chi_2, \chi_3]\!] \in \mathcal{K}_S \qquad \forall\ \Phi_i \in \mathcal{K},\ \chi_i \in \mathcal{K}_S\ . \tag{3.3b}$$

JHEP12(2008)037

Therefore, with the above it is immediate to see that if we shift an element of $\mathcal{K}$ by an arbitrary element in $\mathcal{K}_S$, the part of the resulting bracket which resides in $\mathcal{K}$ does not change. In this sense (3.3) defines the notion of *relaxed closure* over $\mathcal{K}$.

It will be convenient to introduce the notion of "physical" part of a given three-bracket. Let Υ_i be a general element in $\mathcal{K} \oplus \mathcal{K}_S$. It can then be decomposed into its *physical* part Φ_i (which is in $\mathcal{K}$) and its *spurious* part χ_i (which is in $\mathcal{K}_S$). In other words,

$$(\Upsilon_i)_{\text{phys}} = \Phi_i = h_{ab} T^a \; Tr(T^b \Upsilon_i), \quad \forall \Upsilon_i \in \mathcal{K} \oplus \mathcal{K}_S \,, \tag{3.4}$$

where h_{ab} is the inverse of the metric $h^{ab} = Tr(T^a T^b)$. It is also useful to note that

$$Tr(\Phi\chi) = 0 \,, \qquad \forall \Phi \in \mathcal{K}, \; \chi \in \mathcal{K}_S \,, \tag{3.5}$$

and,

$$\big([\![\Upsilon_1, \Upsilon_2, \Upsilon_3]\!]\big)_{\text{phys}} = \big([\![\Phi_1, \Phi_2, \Phi_3]\!]\big)_{\text{phys}} = f^{abc}{}_d \; \Phi_{1a} \Phi_{2b} \Phi_{3c} T^d \,. \tag{3.6}$$

In terms of the physical part of a bracket, the relaxed closure condition is nothing but the closure for the physical part of the brackets.

In the same spirit as above one may define a notion of *relaxed* fundamental identity, by demanding the fundamental identity (2.3) to hold for the physical part of the three brackets. Explicitly,

$$\begin{aligned} &[\![[\![\Upsilon_i, \Upsilon_j, \Upsilon_k]\!]_{\text{phys}}, \Upsilon_l, \Upsilon_m]\!]_{\text{phys}} + [\![[\![\Upsilon_i, \Upsilon_j, \Upsilon_l]\!]_{\text{phys}}, \Upsilon_m, \Upsilon_k]\!]_{\text{phys}} + \\ &\qquad [\![[\![\Upsilon_i, \Upsilon_j, \Upsilon_m]\!]_{\text{phys}}, \Upsilon_k, \Upsilon_l]\!]_{\text{phys}} = [\![\Upsilon_i, \Upsilon_j, [\![\Upsilon_k, \Upsilon_l, \Upsilon_m]\!]_{\text{phys}}]\!]_{\text{phys}}. \end{aligned} \tag{3.7}$$

In terms of the structure constants f, this is equivalent to requiring $f^{abc}{}_d$ to satisfy (2.9).

With above notion of the relaxed closure and fundamental identity, together with the orthogonality properties (3.1), we define a *relaxed-three-algebra* ($\mathcal{RA}_3$). Any given $\mathcal{RA}_3$ has a *physical* part $\mathcal{K}$ and an *spurious* part $\mathcal{K}_S$.

Let us now rewrite the BLG theory with the above *relaxed-three-algebra* by adding T^A components to the physical fields, i.e. we take the fields to be

$$\Upsilon = \Phi_a T^a + \chi_A T^A \,, \tag{3.8}$$

and let the gauge fields to also have $A_{i\;aA}$ components. With the trace conditions (3.1) it is readily seen that the χ_A components of the fields do not appear in the action at all. This is very similar to the notion of physical and spurious states in a 2d CFT e.g. see [23]. Since the action does not involve the spurious fields the equations of motion for the physical fields will not change compared to the ordinary BL case.

One can also check the supersymmetry and the gauge symmetry invariance of the action within the relaxed-three-algebra. The only part which should be checked is where the fundamental identity is used. As discussed in [2] the fundamental identity is needed for the closure of supersymmetry when two successive supersymmetry transformations on the gauge field is considered. One can, however, see that with the structure of the three-brackets introduced in (3.2), the part in equation (35) of [2] does not harm the closure of the supersymmetry algebra as long as $f^{abc}{}_d$ are still satisfying the fundamental identity (2.9).

In the next section we will give a construction based on usual matrices which realizes this relaxed-three-algebras $\mathcal{RA}_3$.

JHEP12(2008)037

4. Matrix representation for the relaxed-three-algebras

There are many three-algebras, already among the ones having a bi-invariant metric with Euclidean and Lorentzian signatures, and one may wonder whether by introducing some additional structure in the theory reviewed above, one may get stronger constraints on the classical Lie algebras underlying them. Inspired by the ideas of [12] regarding quantization of Nambu three-brackets using four-brackets, we propose to realize the three-bracket in terms of a four-bracket:

$$[\![A,B,C]\!] \equiv [\hat{A},\hat{B},\hat{C},T^-] \tag{4.1}$$

where the hatted quantities are just normal matrices, T^- being among them (to be specified shortly) and the four-bracket is defined as

$$\begin{aligned}[\hat{A}_1,\hat{A}_2,\hat{A}_3,\hat{A}_4] &= \frac{1}{4!}\epsilon^{ijkl}\hat{A}_i\hat{A}_j\hat{A}_k\hat{A}_l \\ &= \frac{1}{4!}\Big(\{[\hat{A}_1,\hat{A}_2],[\hat{A}_3,\hat{A}_4]\} - \{[\hat{A}_1,\hat{A}_3],[\hat{A}_2,\hat{A}_4]\} + \{[\hat{A}_1,\hat{A}_4],[\hat{A}_2,\hat{A}_3]\}\Big)\,.\end{aligned} \tag{4.2}$$

The fundamental identity (2.3) in terms of the four-bracket takes the form[1]

$$\begin{aligned}&[[A,B,C,T^-],D,E,T^-] + [C,[A,B,D,T^-],E,T^-] \\ &\qquad + [C,D,[A,B,E,T^-],T^-] = [A,B,[C,D,E,T^-],T^-].\end{aligned} \tag{4.3}$$

It is straightforward to see that the above four-bracket defines a non-associative structure over the algebra of matrices and the Trace over the matrices is the natural trace operation over this algebra.[2] The Hermitian conjugation of the underlying algebra structure naturally extends to the four-bracket. If T^- is Hermitian it is immediate to see that (2.6) holds. As we will show for one of the only two possibilities for T^-, T^- is Hermitian.

In the rest of this section we show that the above proposal (4.1), within the setup of the *relaxed-three-algebras* of previous section, works for the two currently recognized three-algebras, namely the $so(4)$-based algebras [15] and those coming with a Lorentzian signature metrics of [16, 17, 19]. In fact, within our working assumptions described below, these are the only two possible cases.

From the definition it is directly seen that the four-bracket has the anti-symmetry property (2.2). Using the explicit definition (4.2) and standard matrix algebra, it is straightforward to see that the "by-part integration" property (2.5) is also satisfied. We are then left with verifying the (relaxed) closure and fundamental identities.

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[1] For the ease of notation, we will omit the hats $\hat{A}$ on any matrix A. It should be clear from the bracket under consideration the nature of the object under consideration.

[2] What we mean by lack of associativity of the four-bracket structure is what has also been called (lack of) Leibniz rule (e.g see [11]):

$$[A,B,CD,T] \neq [A,B,C,T]D + C[A,B,D,T]\ .$$

4.1 The relaxed closure and fundamental identities

All the elements we consider belong to a finite dimensional matrix representation of an underlying Lie-algebra $\mathcal{G}$. $\mathcal{G}$ is an ordinary (classical) Lie-algebra defined through commutator relations and ordinary structure constants. The three-bracket structure is, however, defined over a subset of $\mathcal{G}$. This subset has two parts: $\mathcal{K}$ with the basis T^a, and $\mathcal{K}_S$ with the basis T^A. $\mathcal{K}$ contains the "physical fields" and $\mathcal{K}_S$ the "spurious fields" (*cf.* discussions of section 3). We should emphasize that, although both $\mathcal{K}$ and $\mathcal{K}_S$ are subsets of $\mathcal{G}$ they are not necessarily sub-algebras of $\mathcal{G}$.

The *relaxed* closure conditions (3.3) for three-algebras within our four-bracket structure are then written as

$$[\Phi_1, \Phi_2, \Phi_3, T^-] \in \mathcal{K} \oplus \mathcal{K}_S, \qquad \forall \Phi_i \in \mathcal{K}\,, \tag{4.4a}$$

$$[\Phi_1, \Phi_2, \chi, T^-],\ [\Phi_1, \chi_1, \chi_2, T^-],\ [\chi_1, \chi_2, \chi_3, T^-] \in \mathcal{K}_S \qquad \forall \Phi_i \in \mathcal{K},\ \chi_i \in \mathcal{K}_S\ . \tag{4.4b}$$

In fact we can view the above closure conditions as the definitions for the subsets $\mathcal{K}$ and $\mathcal{K}_S$ in $\mathcal{G}$.

For the relaxed-three-algebras $\mathcal{RA}_3$ we demand a relaxed version of the fundamental identity (3.7). Namely, we only demand *the non-spurious part of the brackets in (4.4a) to satisfy the fundamental identity.*

Since T^- has a distinct role in our four-bracket construction, we must specify it separately. From the closure conditions (4.4) and the definition of the four-bracket (4.2) it is evident that T^- is either in $\mathcal{K}$ or $\mathcal{K}_S$. To obtain a non-trivial interacting theory, T^- cannot be in $\mathcal{K}_S$. This can be seen by recalling the trace conditions (3.1) on the spurious parts. Thus we take T^- to be in $\mathcal{K}$.

To proceed we will choose T^- to be an element of $\mathcal{K}$, such that its *anti-commutator* with any element of $\mathcal{K}$ and $\mathcal{K}_S$ is in the center of the underlying algebra $\mathcal{G}$, as our *working assumptions*. In terms of the basis T^a and T^A this means that either T^- anticommutes with T^a and T^A, or its anti-commutator with them is the identity matrix:

$$\begin{aligned} &\{T^-, T^a\} = 0, \quad \text{or} \quad \{T^-, T^a\} = \mathbb{1}. \\ &\{T^-, T^A\} = 0. \end{aligned} \tag{4.5}$$

(Note that $\{T^-, T^A\} = \mathbb{1}$ case is not possible due to the trace condition (3.1).)

With the above choice it is evident that any linear combination of a given set of T^a's is also satisfying the above anti-commutator conditions. Therefore, within the set of T^a's one can identify a *single* element whose anti-commutator with T^- is the identity matrix. We will denote this element by T^+. As $T^- \in \mathcal{K}$, T^- should then square to zero or to $(1/2)\mathbb{1}$. Hence, given our working assumptions, there are two cases to consider for our four-bracket realization of the three-bracket:

i) $T^- = T^+$, corresponding to $2(T^-)^2 = \mathbb{1}$.

ii) $(T^-)^2 = 0$, corresponding to $T^+ \neq T^-$ and $\{T^+, T^-\} = \mathbb{1}$.

JHEP12(2008)037

We will denote the elements in $\mathcal{K}$ by $\{T^a\} = \{T^+,\ T^-, T^\alpha\}$. Without loss of generality, one can always choose the basis such that

$$\{T^\pm, T^\alpha\} = 0, \qquad \{T^+, T^-\} = \mathbb{1}\ . \tag{4.6}$$

Note that while $\{T^+, T^A\}$ can be non-vanishing, it is always traceless (*cf.* (3.1)).

We will choose the T^α matrices to be hermitian,

$$(T^\alpha)^\dagger = T^\alpha\,, \tag{4.7}$$

therefore, the metric

$$h^{\alpha\beta} = Tr(T^\alpha T^\beta),$$

is positive definite. Recalling (4.6),

$$h^{\pm\alpha} = Tr(T^\pm T^\alpha) = 0. \tag{4.8}$$

Thus, the $T^- = T^+$ case corresponds to a positive definite metric h^{ab} since $2(T^-)^2 = \mathbb{1}$, and consequently h^{--} is positive. On the other hand, the $T^- \neq T^+$ case has Lorentzian signature. This is because $\{T^-, T^+\} = \mathbb{1}$, and so $h^{-+} = h^{+-}$ is positive definite and $h^{--} = 0$. Hence

$$\det h_{ab} = -\det h_{\alpha\beta} \cdot (h^{+-})^2 < 0. \tag{4.9}$$

One can always find a linear combination of T^+ and T^- for which both h^{--} and h^{++} vanish. Here we choose to work in such a basis.

Equipped with the above we are now ready to examine the relaxed closure condition (4.4) and the fundamental identity and check which algebras are satisfying the above requirements.

4.1.1 The Euclidean signature case

For this case the only non-vanishing "physical" four-bracket is of the form $[T^\alpha, T^\beta, T^\gamma, T^-]$ which recalling (4.5) can be written as

$$[T^\alpha, T^\beta, T^\gamma, T^-] = F^{\alpha\beta\gamma} T^-\,, \tag{4.10}$$

where $F^{\alpha\beta\gamma}$ is the totally anti-symmetric three-form

$$F^{\alpha\beta\gamma} = \frac{1}{12}\Big(\{T^\alpha, [T^\beta, T^\gamma]\} + \{T^\gamma, [T^\alpha, T^\beta]\} + \{T^\beta, [T^\gamma, T^\alpha]\}\Big)\,. \tag{4.11}$$

Note that by definition $F^{\alpha\beta\gamma}$ is not necessarily in the algebra $\mathcal{G}$, but in general in its enveloping algebra.

The relaxed closure condition (4.4) demands $F^{\alpha\beta\gamma}T^- \in \mathcal{K} \oplus \mathcal{K}_S$. Equivalently,

$$[T^\alpha, T^\beta, T^\gamma, T^-] = f^{\alpha\beta\gamma}{}_\lambda T^\lambda + g^{\alpha\beta\gamma} T^- + k^{\alpha\beta\gamma}{}_A T^A \tag{4.12}$$

where $f,\ g$ and k are expansion coefficients, anti-symmetric in $\alpha\beta\gamma$ indices.

Multiplying both sides of (4.12) with T^- and taking trace of both sides implies that $g^{\alpha\beta\gamma} = 0$ and hence we only remain with $f^{\alpha\beta\gamma}{}_\lambda$ and $k^{\alpha\beta\gamma}{}_A$ terms.

The relaxed fundamental identity (3.7) then requires:

$$f^{\alpha\beta\gamma}{}_{\sigma}\, f^{\sigma\rho\lambda}{}_{\delta} + f^{\alpha\beta\rho}{}_{\sigma}\, f^{\gamma\sigma\lambda\delta} + f^{\alpha\beta\lambda}{}_{\sigma}\, f^{\gamma\rho\sigma}{}_{\delta} = f^{\gamma\rho\lambda}{}_{\sigma}\, f^{\alpha\beta\sigma}{}_{\delta}, \tag{4.13}$$

Since h^{ab} is positive definite, it was proved in [15] that the *unique* solution to (4.13) is given by

$$f^{\alpha\beta\gamma\rho} = \epsilon^{\alpha\beta\gamma\rho} \tag{4.14}$$

and $\alpha, \beta, \gamma, \rho = 1, 2, 3, 4$. The explicit solution for this case, as has been discussed in [12, 24] is

$$T^{\alpha} = \mathcal{J}^{\alpha}, \qquad T^{-} = \mathcal{L}_5 \tag{4.15}$$

where $\mathcal{J}^{\alpha}$ and $\mathcal{L}_5$ are in general $2J \times 2J$ representation of $so(4)$, which are generalization of the ordinary SO(4) Dirac gamma matrices [24].[3] For $J = 2$ they reduce to γ^{α} and γ^5. (For an explicit matrix form and more detailed discussion see [12, 24].) Note that the size of the representation is not fixed by the above considerations.

The above explicit representation for T^{α}'s leads to $k^{\alpha\beta\gamma}{}_{A} = 0$ and hence for this case, the Euclidean case, the $\mathcal{RA}_3$ is the same as the corresponding ordinary BL three-algebra.

In summary, our four-bracket representation for the three-algebra and its three-bracket has all the needed properties of the three-bracket and the only solution to this case is the SO(4)-based solutions discussed in [15].

Finally it is notable that in this case the algebra $\mathcal{G}$ which is the algebra generated from $\mathcal{J}^{\alpha}$ and $\mathcal{L}_5$ (and their commutators) is $so(6) \simeq su(4)$. Note, however, that the $\mathcal{J}^{5\alpha} = i[\mathcal{J}^{\alpha}, \mathcal{L}_5]$ are not the T^A's, as they do not satisfy the trace condition (3.1) and (3.2b).

4.1.2 The Lorentzian signature case

There are two different non-vanishing four-brackets of "physical" elements to consider:

$$[T^{\alpha}, T^{\beta}, T^{\gamma}, T^{-}] = F^{\alpha\beta\gamma} T^{-} \tag{4.16a}$$

$$[T^{\alpha}, T^{\beta}, T^{+}, T^{-}] = \frac{1}{4}\,[T^{\alpha}, T^{\beta}] T\,, \tag{4.16b}$$

where $F^{\alpha\beta\gamma}$ is defined in (4.11) and

$$T \equiv [T^{+}, T^{-}]. \tag{4.17}$$

In deriving these expressions, we have used the fact that, by definition, T commutes with T^{α} ($[T, T^{\alpha}] = 0$). Furthermore, from $(T^{-})^2 = 0$ and $\{T^{+}, T^{-}\} = \mathbb{1}$, one has

$$T^2 = -(\mathbb{1} - 2T^{+}T^{-})(\mathbb{1} - 2T^{-}T^{+}) = \mathbb{1}, \tag{4.18a}$$

$$TT^{-} = -(\mathbb{1} - 2T^{+}T^{-})T^{-} = -T^{-}\ . \tag{4.18b}$$

Let us analyze the (relaxed) closure conditions. First, requiring $[T^{\alpha}, T^{\beta}, T^{\gamma}, T^{-}] \in \mathcal{K} \oplus \mathcal{K}_S$ implies that in the most general form

$$F^{\alpha\beta\gamma} T^{-} = f^{\alpha\beta\gamma}{}_{\lambda} T^{\lambda} + g^{\alpha\beta\gamma} T^{-} + l^{\alpha\beta\gamma} T^{+} + k^{\alpha\beta\gamma}{}_{A} T^{A} \tag{4.19}$$

[3] As mentioned in [15] direct sums of an arbitrary $so(4)$ algebras also leads to $f^{\alpha\beta\gamma\rho} = \epsilon^{\alpha\beta\gamma\rho}$.

JHEP12(2008)037

where f, g, k and l are some unknown arbitrary expansion parameters which are totally anti-symmetric under exchange of α, β and γ indices. Multiplying both sides of (4.19) in T^- and taking the trace, noting that the left-hand-side vanishes identically, we learn that $l^{\alpha\beta\gamma} = 0$. Noting that $(T^-)^2 = 0$ and $\{T^-, F^{\alpha\beta\gamma}\} = 0$ then

$$\{T^-, F^{\alpha\beta\gamma}T^-\} = 0, \qquad [T^-, F^{\alpha\beta\gamma}T^-] = 0,$$

and therefore[4]

$$T^- \left(f^{\alpha\beta\gamma}{}_\lambda T^\lambda + k^{\alpha\beta\gamma}{}_A T^A\right) = 0. \tag{4.20}$$

The second relaxed closure requirement, $[T^\alpha, T^\beta, T^+, T^-] \in \mathcal{K} \oplus \mathcal{K}_S$ implies,

$$[T^\alpha, T^\beta, T^+, T^-] = \frac{1}{4} f^{\alpha\beta}{}_\gamma T^\gamma + l^{\alpha\beta}T^- + g^{\alpha\beta}T^+ + k^{\alpha\beta}{}_A T^A \tag{4.21}$$

where $f^{\alpha\beta}{}_\gamma$, $l^{\alpha\beta}$, $g^{\alpha\beta}$, $k^{\alpha\beta}{}_A$ are some unknown coefficients to be determined later. Taking anti-commutator of both sides of (4.21) with T^- we learn that coefficient of T^+ is zero, $g^{\alpha\beta} = 0$. Multiplying both sides with T^+ and taking the trace (recall (4.16)) the left hand side vanishes and therefore $l^{\alpha\beta} = 0$. Commutator of both sides of (4.21) with T, leads to $k^{\alpha\beta}{}_A[T, T^A] = 0$. On the other hand if we multiply both sides of (4.21) with T^- and then its commutator with T^+ we learn that $k^{\alpha\beta}{}_A(2T^A + [T, T^A]) = 0$ and hence $k^{\alpha\beta}{}_A T^A = 0$.

Using the above and in particular $[T^\alpha, T^\beta] = f^{\alpha\beta}{}_\gamma TT^\gamma$, that $TT^- = -T^-$ and that $Tr(T^\alpha T^-) = 0$ one can show that trace of any number of T^α's with T^- is zero. This in particular implies that $f^{\alpha\beta\gamma}{}_\lambda h^{\lambda\rho} = 0$. $h^{\alpha\beta}$ is non-degenerate and invertible therefore,

$$f^{\alpha\beta\gamma}{}_\lambda = 0\,, \tag{4.22}$$

and (4.20) reduces to $k^{\alpha\beta\gamma}{}_A T^A T^- = 0$ and moreover we have

$$\{T^+, F^{\alpha\beta\gamma}T^-\} = -F^{\alpha\beta\gamma}\, T = g^{\alpha\beta\gamma}\, \mathbb{1} + k^{\alpha\beta\gamma}{}_A \{T^+, T^A\}\ . \tag{4.23}$$

After the above analysis in summary we remain with

$$[T^\alpha, T^\beta, T^\gamma, T^-] = F^{\alpha\beta\gamma}T^- = g^{\alpha\beta\gamma}T^- + k^{\alpha\beta\gamma}{}_A T^A \tag{4.24a}$$

$$[T^\alpha, T^\beta, T^+, T^-] = \frac{1}{4}\,[T^\alpha, T^\beta]T = f^{\alpha\beta}{}_\gamma T^\gamma\,, \tag{4.24b}$$

Furthermore, using (2.5) we learn that

$$f^{\alpha\beta\gamma} = f^{\alpha\beta}{}_\rho h^{\rho\gamma} = -\frac{1}{2} Tr(\mathbb{1}) g^{\alpha\beta\gamma}\ . \tag{4.25}$$

To complete our analysis and to determine the yet unknown coefficients $k^{\alpha\beta\gamma}{}_A$ and $f^{\alpha\beta}{}_\gamma$ we examine the relaxed fundamental identity. Let us first rewrite the identity for generic generators T^a:

$$\begin{aligned}&[[T^a, T^b, T^c, T^-]_{\rm phys}, T^d, T^e, T^-]_{\rm phys} + [T^c, [T^a, T^b, T^d, T^-]_{\rm phys}, T^e, T^-]_{\rm phys} +\\ &+ [T^c, T^d, [T^a, T^b, T^e, T^-]_{\rm phys}, T^-]_{\rm phys} = [T^a, T^b, [T^c, T^d, T^e, T^-]_{\rm phys}, T^-]_{\rm phys},\end{aligned} \tag{4.26}$$

[4]Here we will assume working with the non-trivial case of $T^\alpha T^- \neq 0$.

where $T^a = T^\alpha, T^-$ or T^+. For three choices of the $(abcde)$ indices the above fundamental identity does not trivially hold, these cases are:
i) $(abcde) = (\alpha\beta\gamma\rho+)$ implying

$$f^{\alpha\beta}{}_\sigma g^{\gamma\rho\sigma} = f^{\gamma\rho}{}_\sigma g^{\alpha\beta\sigma} \ . \tag{4.27}$$

ii) $(abcde) = (\alpha + \gamma\rho\lambda)$ implying

$$f^{\alpha\gamma}{}_\sigma g^{\rho\lambda\sigma} + f^{\alpha\rho}{}_\sigma g^{\lambda\gamma\sigma} + f^{\alpha\lambda}{}_\sigma g^{\gamma\rho\sigma} = 0. \tag{4.28}$$

iii) $(abcde) = (\alpha + \gamma\rho+)$ implying

$$f^{\alpha\gamma}{}_\sigma f^{\rho\sigma}{}_\lambda + f^{\rho\alpha}{}_\sigma f^{\gamma\sigma}{}_\lambda + f^{\gamma\rho}{}_\sigma f^{\alpha\sigma}{}_\lambda = 0. \tag{4.29}$$

Recalling (4.25) the only independent of the above equations is (4.29).

Noting (4.16b), (4.21) and that $[T, T^\alpha] = 0$, it is seen that

$$[TT^\alpha, TT^\beta] = f^{\alpha\beta}{}_\gamma TT^\gamma, \tag{4.30}$$

therefore, recalling (4.29), TT^α's are generators of a (classical) Lie-algebra which is a sub-algebra of $\mathcal{G}$, with the structure constants $f^{\alpha\beta}{}_\gamma$. We will denote this sub-algebra by $\mathcal{H}$.

Given any classical Lie algebra $\mathcal{H}$, the only remaining parameter in our brackets is $k^{\alpha\beta\gamma}{}_A$. As discussed

$$k^{\alpha\beta\gamma}{}_A T^A T^- = 0\,, \tag{4.31}$$

which can only be satisfied if either $k^{\alpha\beta\gamma}{}_A$ or $T^A T^- = 0$. The first choice is not a possibility, because there is no classical Lie-algebra other than $su(2)$ for which the totally anti-symmetric three tensor

$$F^{\alpha\beta\gamma} T = \frac{1}{12}\Big(\{TT^\alpha, [TT^\beta, TT^\gamma]\} + \{TT^\gamma, [TT^\alpha, TT^\beta]\} + \{TT^\beta, [TT^\gamma, TT^\alpha]\}\Big)\ ,$$

is proportional to the identity.[5] So, we are forced to choose the other possibility, i.e.

$$T^A T^- = 0 \quad \Rightarrow \quad TT^A = -T^A T = -T^A. \tag{4.32}$$

We may solve the above as

$$T^A = T^- \tilde{T}^A = \tilde{T}^A T^- \quad \Longleftrightarrow \quad \tilde{T}^A = \{T^+, T^A\}\,, \tag{4.33}$$

where

$$[T^\pm, \tilde{T}^A] = [T, \tilde{T}^A] = 0\ . \tag{4.34}$$

In terms of $\tilde{T}^A$, (4.23) is written as

$$-F^{\alpha\beta\gamma}\ T = g^{\alpha\beta\gamma}\ \mathbb{1} + k^{\alpha\beta\gamma}{}_A \tilde{T}^A\ . \tag{4.35}$$

To elaborate on the spurious sector and in particular the algebra of the $\tilde{T}^A$'s, we examine the relaxed closure condition for the brackets involving T^A. As it is seen from (4.4b)

[5]We will return to the special case of $\mathcal{H} = su(2)$ later in this section.

JHEP12(2008)037

there are three such cases. For brackets of the form $[T^a, T^b, T^A, T^-]$ when both a and b are α-type the bracket vanishes and the only non-vanishing case is when $(ba) = (+\alpha)$. After some algebra we find

$$12[T^\alpha, T^A, T^+, T^-] = [TT^\alpha, T^A] = C^{\alpha A}{}_B T^B\,, \tag{4.36}$$

where the second equality is the statement of relaxed closure, with some unknown constants C. From the above we also have

$$[TT^\alpha, \tilde{T}^A] = C^{\alpha A}{}_B \tilde{T}^B\ . \tag{4.37}$$

Consistency of the above equation implies that

$$C^{\alpha A}{}_B C^{\beta B}{}_D - C^{\beta A}{}_B C^{\alpha B}{}_D = f^{\alpha\beta}{}_\gamma C^{\gamma A}{}_D\ . \tag{4.38}$$

Since $T^A T^B = 0$, brackets involving two and three T^A's identically vanish. Given the above information we now proceed to construct the underlying algebra $\mathcal{G}$.

The algebra $\mathcal{I}$ constructed from T, T^+ and T^-. As discussed above (4.17), (4.18)

$$[T, T^-] = -2T^-, \quad [T^+, T^-] = T, \quad \{T^+, T^-\} = \mathbb{1}.$$

Thus, $\mathcal{I}$ will be identified once $[T, T^+]$ is known. It is straightforward to show

$$\{[T, T^+], T^-\} = 2\cdot\mathbb{1}, \quad \{[T, T^+], T^\alpha\} = 0, \quad [[T, T^+], T^-]] = 2T,$$

and hence $[T^\alpha, T^\beta, [T, T^+], T^-] = 12[T^\alpha, T^\beta]T \in \mathcal{K}$. Therefore, $[T, T^+]$ is an element in $\mathcal{K}$, and since its anti-commutator with T^- equals $2\cdot\mathbb{1}$, we conclude

$$[T, T^+] = +2T^+. \tag{4.39}$$

This is also consistent with all other properties quoted above. As a consequence, one can show that $(T^+)^2 = 0$.

To sum up, T, T^+ and T^- form the following algebra

$$[T, T^\pm] = \pm 2T^\pm, \qquad [T^+, T^-] = T \tag{4.40a}$$
$$\{T^+, T^-\} = \mathbb{1} \qquad (T^-)^2 = 0. \tag{4.40b}$$

Equations (4.40a) fix the algebra to be $su(2)$ while (4.40b) fixes its representation to be 2×2 matrices. An explicit solution to the above equations is

$$T^- = \sigma^- = \frac{1}{2}(\sigma^1 - i\sigma^2), \quad T = \sigma^3, \quad T^+ = \sigma^+ = \frac{1}{2}(\sigma^1 + i\sigma^2), \tag{4.41}$$

where σ^i are the Pauli matrices. It is also noteworthy that $(T^+)^\dagger = T^-$, $T^\dagger = T$.

Fixing the underlying algebra $\mathcal{G}$. The algebra $\mathcal{G}$ is obtained by studying the closure of the commutators between the generators of its $\mathcal{I}$ and $\mathcal{H}$ sub-algebras as well as the algebra constructed from $\tilde{T}^A$'s, which will be denoted by $\tilde{\mathcal{H}}$. With the above considerations (the commutator or) the algebra of $\tilde{T}^A$'s will not be fixed. However, from (4.37) and (4.35) it is seen that $\tilde{\mathcal{H}}$ should contain $\mathcal{H}$ as a subalgebra. Moreover, in general $\tilde{\mathcal{H}}$ may be taken as the enveloping algebra of $\mathcal{H}$, $Env(\mathcal{H})$, or depending on $\mathcal{H}$, some particular subalgebra of $Env(\mathcal{H})$.

To complete our analysis it will be useful to give an explicit representation for the underlying algebra $\mathcal{G}$. Based on what we have discussed any element in $\mathcal{G}$, and in particular T^α, $\tilde{T}^A$, $T^\pm$ and T can be written as

$$\begin{aligned} T^\pm &= \mathbb{1} \otimes \sigma^\pm, \qquad & T &= \mathbb{1} \otimes \sigma^3 \\ T^\alpha &= t^\alpha \otimes \sigma^3, \qquad & \tilde{T}^A &= \tilde{t}^A \otimes \sigma^3\,, \end{aligned} \tag{4.42}$$

where t^α and $\tilde{t}^A$ are respectively generators of $\mathcal{H}$ and $\tilde{\mathcal{H}}$, that is

$$[t^\alpha, t^\beta] = f^{\alpha\beta}{}_\gamma\, t^\gamma\,, \qquad [t^\alpha, \tilde{t}^A] = C^{\alpha A}{}_B\, \tilde{t}^B\ . \tag{4.43}$$

As discussed neither of the algebras $\mathcal{H}$ and $\tilde{\mathcal{H}}$ nor their representations are fixed. However, as a general solution one may take $\tilde{\mathcal{H}} = Env(\mathcal{H})$ (up to an Abelian $u(1)$ factor) in which case, if we choose to work with $N \times N$ representation of $\mathcal{H}$ the algebra $\tilde{\mathcal{H}}$, independently of $\mathcal{H}$, will be $su(N)$ and therefore $\mathcal{G} = su(2N)$ (see, however, the comment below). A special case which is physically well-motivated is $\mathcal{H} = \tilde{\mathcal{H}}$. For this case $\mathcal{H}$ is necessarily fixed to be $su(N)$ in its *fundamental* $N \times N$ representation. In this case

$$g^{\alpha\beta\gamma} = -\frac{1}{N} f^{\alpha\beta\gamma}$$

where f are the structure constants of $su(N)$.

Before closing this section, three comments are in order:

- For the very special case of $\mathcal{H} = su(2)$ and in its fundamental 2×2 representation, it is readily seen that one can take $k^{\alpha\beta\gamma}{}_A$ to be zero. In this case there is no need to introduce the spurious sector $\mathcal{K}_S$. Nonetheless, for this case again the underlying algebra $\mathcal{G}$ will be $su(4)$.

- Although we usually consider $\mathcal{H}$ to be a simple Lie-algebra, it could also be a semi-simple algebra. The particular and interesting example of this case is $\mathcal{H} = so(4)$. (Note, however, that as discussed above this is not the Euclidean three-algebra.) In this case, if we work with the 4×4 representation of $so(4)$ algebra then we can take $\tilde{H} = so(4) \times u(1) \times u(1)$ in which case the two $u(1)$ factors are generated by γ^5 and the 4×4 identity matrix. With this choice the consistency relation (4.38) is obviously satisfied. The underlying algebra $\mathcal{G}$ in this case is 8×8 representation of $su(4) \times su(4)$.

- As we have discussed the underlying algebra in both of the $(T^-)^2 = \mathbb{1}/2$ and $(T^-)^2 = 0$ cases can be (and indeed for the physically interesting ones is) an $su(2N)$ algebra.

JHEP12(2008)037

The $N = 2$ case, related to the $\mathcal{G} = su(4)$, is very special because it is isomorphic to $so(6)$. In this sense it may seem that for the $\mathcal{G} = su(4)$ case there are two different (Euclidean and Lorentzian) solutions. But, it turns out that both of these solutions are indeed physically the same and they are related by a change of basis T^a's: take $\mathcal{H} = su(2)$ and choose $T^+ - T^- = \mathbb{1}_2 \otimes i\sigma^2$ as $i\gamma^5$ and the four T^a's (the Dirac γ matrices) to be $2T^\alpha = \sigma^\alpha \otimes \sigma^3$, $\alpha = 1, 2, 3$ and $T^+ + T^- = \mathbb{1}_2 \otimes \sigma^1$. One should, however, note that such a change of basis and taking the linear combination of generators as new "T^-" does not generally work because the fundamental identity is not linear in T^-. It is not difficult to show, using direct examination of the fundamental identity, that it only works for $\mathcal{H} = su(2)$. As a related comment, we note that in the $(T^-)^2 = \mathbb{1}/2$ case T^- is hermitian and in the $(T^-)^2 = 0$ it is not. As we have discussed the $(T^-)^2 = 0$ case does not have a solution with hermitian T^-.

5. The alternative representation for the BL theory

After replacing the BL three-algebras with the relaxed-three-algebras $\mathcal{RA}_3$ and realizing the relaxed-three-brackets with the four-brackets of usual matrices, we are now ready to re-write the BL action in terms of usual matrices; the only thing we need to do is to replace the three-brackets of the BL action with the four-bracket and recall the definition of the trace. As discussed we take our gauge fields to have $A_{i\ ab}$ and $A_{i\ aA}$ components and define the covariant derivative of any element Φ in $\mathcal{K} \oplus \mathcal{K}_S$ as

$$D_i\Phi = \partial_i\Phi - [T^a, T^b, \Phi, T^-]A_{i\ ab} - [T^a, T^A, \Phi, T^-]A_{i\ aA}\ . \tag{5.1}$$

As shown the spurious parts of the field Φ as well as its covariant derivative do not appear in the action (as they drop out once we take the trace). Therefore, we can define a "physical gauge" in which $\Phi_A = 0$ and A_{iaA} components are chosen such that[6]

$$D_i\Phi = (D_i\Phi)_{\text{phys}} = \partial_i\Phi_{\text{phys}} - A_{i\ ab}[T^a, T^b, \Phi, T^-]_{\text{phys}} = \left(\partial_i\Phi_d - f^{\text{abc}}{}_d A_{i\ ab}\Phi_c\right) T^d\ . \tag{5.2}$$

Equivalently, the "physical gauge" is the one in which $\Phi,\ D_i\Phi \in \mathcal{K}$. As discussed in the Lorentzian case, when we choose TT^α to be Hermitian matrices, then T^A's are not Hermitian. Therefore in the physical gauge, when T^A components are absent we can demand Hermiticity

$$\Phi^\dagger = \Phi\,, \qquad (D_i\Phi)^\dagger = D_i\Phi\,, \tag{5.3}$$

where Φ are generic scalar fields of the theory. In fact we will be requiring the above conditions which also implies working with non-spurious parts of fields. Hereafter we will always be working in the above mentioned physical gauge (5.3) and unless it is necessary this point will not be mentioned explicitly. Therefore, in the physical gauge the $A_{i\ aA}$ components do not appear and we only remain with $A_{i\ ab}$ components of the gauge field.

[6]Note that due to the possibility of the presence of $A_{i\ aA}$ components we have an extended notion of gauge symmetry which allows for choosing these components of the gauge fields. Since these components do not appear in the Chern-Simons part of the action, this gauge symmetry is of course a trivial symmetry of the corresponding BL action.

JHEP12(2008)037

Next we focus on the $A_{i\ ab}$ components. In general, A_{ab} can have $A_{-\alpha}$ and $A_{\alpha\beta}$ components for the $so(4)$-based algebra and $A_{\pm\mp}$, $A_{\pm\alpha}$ and $A_{\alpha\beta}$ components for the Lorentzian algebras. However, as it is seen from the explicit form of the covariant derivative (5.1) and also the form of the twisted Chern-Simons action (2.13), not all of the possible components of the gauge field appear in the action. For the $so(4)$ based algebra it is only the $\alpha\beta$ component [1, 2], and for the Lorentzian case they are the $+\alpha$ and $\alpha\beta$ components [17, 16]. With the above definition, hence the other components, i.e. $A_{-\alpha}$ for the $so(4)$-based case and $A_{\pm\mp}$ and $A_{-\alpha}$ for the Lorentzian case, are "gauge degrees of freedom" and may be chosen freely and for example can be set to zero. It is also seen that the T^- component of the $\Phi \in \mathcal{K}$, for both the Euclidean and Lorentzian cases, is also a free field not interacting with the other components.

5.1 Lagrangian in terms of Four-brackets

From the discussions of previous section and our construction of three-brackets and the relaxed-three-algebras it is evident that if in the action (2.12) we replace three-brackets with our prescribed four-brackets we will obtain a supersymmetric and gauge invariant action. For both cases, Euclidean and Lorentzian, the supersymmetry transformations and Lagrangian are alike. For completeness we only show the explicit form of the action, its equations of motion and supersymmetry and gauge transformations.

The action.

$$\begin{aligned}S = \int d^3\sigma\ Tr\Big[&-\frac{1}{2}D_iX^ID^iX^I - \frac{1}{2.3!}[X^I,X^J,X^K,T^-][X^I,X^J,X^K,T^-]\\&+\frac{i}{2}\bar{\Psi}\gamma^iD_i\Psi - \frac{i}{4}[\bar{\Psi},X^I,X^J,T^-]\Gamma^{IJ}\Psi\\&+\frac{1}{2}\epsilon^{ijk}\left(A_{i\ ab}\partial_jA_{k\ cd}T^d + \frac{2}{3}A_{i\ ab}A_{j\ de}A_{k\ cf}[T^d,T^e,T^f,T^-]\right)[T^a,T^b,T^c,T^-]\Big].\end{aligned} \tag{5.4}$$

Equations of motion.

$$\begin{aligned}&\left(\gamma^iD_i\Psi + \frac{1}{2}\Gamma^{IJ}[X^I,X^J,\Psi,T^-]\right)_{\text{phys}} = 0\\&\left(D^2X^I - \frac{i}{2}\Gamma^{IJ}[\bar{\Psi},X^J,\Psi,T^-] + \frac{1}{2}[X^I,X^J,[X^I,X^J,X^K,T^-],T^-]\right)_{\text{phys}} = 0\\&\left(\tilde{F}^{ab}_{ij} + \epsilon_{ijk}\big(D^kX^I[T^a,T^b,X^I,T^-] - \frac{i}{2}\bar{\Psi}\gamma^k[T^a,T^b,\Psi,T^-]\big)\right)_{\text{phys}} = 0\end{aligned} \tag{5.5}$$

where $\tilde{F}$ is appeared in (2.18).

Supersymmetry transformations.

$$\begin{aligned}\delta X^I &= i\bar{\epsilon}\Gamma^I\Psi\\\delta\Psi &= D_iX^I\Gamma^I\gamma^i\epsilon - \frac{1}{6}[X^I,X^J,X^K,T^-]\Gamma^{IJK}\epsilon\\\delta(D_i\Phi) - D_i(\delta\Phi) &= i\bar{\epsilon}\gamma^i\Gamma^I[X^I,\Psi,\Phi,T^-], \qquad \forall\Phi\end{aligned} \tag{5.6}$$

where it is understood that we are only considering the physical parts of the fields. It is immediate to see that the action is invariant when we also include non-physical and spurious parts in the above supersymmetry transformations. Nonetheless, along the line of arguments of [2] on can show that the supersymmetry algebra (i.e. commutator of two successive supersymmetry transformations) does not close to a translation, up to gauge transformations.

Gauge transformations. We should emphasize that the following "gauge transformations" are the gauge symmetry remaining after fixing the physical gauge (5.2) and (5.3).

The Euclidean case

$$\begin{aligned} \delta\Phi_a &= \epsilon^{cdb}{}_a \Lambda_{cd}\Phi_b\,, \qquad \delta\Phi_- = 0 \\ \delta A_{i\ ab} &= \partial_i\Lambda_{ab} - \epsilon^{dec}{}_{[a}\Lambda_{b]c}A_{i\ de}\,, \qquad a,b,c,d = 1,2,3,4\ . \end{aligned} \tag{5.7}$$

The Lorentzian case

$$\begin{aligned} \delta\Phi_\alpha &= f^{\beta\gamma}{}_\alpha(2\Lambda_{+\beta}\Phi_\gamma + \Lambda_{\beta\gamma}\Phi_+) \\ \delta\Phi_+ &= f^{\alpha\beta\gamma}\Lambda_{\alpha\beta}\Phi_\gamma \\ \delta\Phi_- &= 0 \end{aligned} \tag{5.8}$$

$$\begin{aligned} \delta A_{i\ +\alpha} &= \partial_i\Lambda_{+\alpha} + 2f^{\beta\gamma}{}_\alpha\Lambda_{+\gamma}A_{i\ +\beta} \\ \delta A_{i\ \alpha\beta} &= \partial_i\Lambda_{\alpha\beta} - 2f^{\rho\gamma}{}_{[\alpha}\ \Lambda_{\beta]\gamma}\ A_{i\ +\rho} - f^{\rho\gamma}{}_{[\alpha}\Lambda_{\beta]+}\ A_{i\ \rho\gamma} \end{aligned} \tag{5.9}$$

In the Lorentzian case the Greek indices are ranging from $1,\cdots,dim\mathcal{H}$ and correspond to $\mathcal{H}$ indices.

5.2 On the physical interpretation of the Lorentzian case

As has been discussed in the literature the $so(4)$-based theories describe (the low energy limit of) two M2-branes on an orbifold [25, 26]. The physical interpretation of the Lorentzian case, however, is less clear. In the *usual treatment* all the components of the scalars X^I, including X^+ and X^- are taken to be real and hence the negative signature in the metric h_{ab} means that one combination of X^+ and X^- has negative eigenvalue, in other words, we have ghosts. Existence of ghosts which couple to the other fields endangers the unitarity of the theory. *Our treatment* of the three-algebras, however, sheds light on the unitarity or ghost problem of the Lorentzian case.

As shown in section 3, the negative eigenvalue of the metric is indeed a reflection of the way we realize the three-brackets and the way $T^\pm$ are embedded in the underlying $\mathcal{G}$ algebra. Therefore, in contrast to the usual treatment in our description, while the scalar field $X^I = X^I{}_aT^a$ is still Hermitian X^+ and X^- are not, explicitly

$$(X^I)^\dagger = X^I \quad \Rightarrow \quad (X^I_+)^* = X^I_-\ . \tag{5.10}$$

With the above it is immediate to see that we do not have the negative kinetic term, or ghost problem. Nonetheless, the unitarity problem shows up in some other place: the interaction terms in the Hamiltonian only involve X^I_+ (and not X^I_-) and hence the Hamiltonian in our description is not Hermitian.

JHEP12(2008)037

To resolve the problem we recall the gauge symmetry of our Lagrangian and the fact that X^I_+ components are *not* gauge invariant (5.8) and hence are not directly physical observables. This opens up the possibility that this non-Hermiticity can be an artifact of the gauge symmetry and the physical theory is indeed Hermitian and unitary. In what follows we argue that there is a gauge, the *Hermitian gauge*, in which the Hamiltonian is explicitly Hermitian, resolving the problem with unitarity.

5.2.1 The Hermitian gauge

As is seen from (5.8) the gauge transformations are parameterized through two sets of gauge parameters $\Lambda_{+\alpha}$ and $\Lambda_{\alpha\beta}$, each having $dim\mathcal{H}$ number of parameters. Moreover, X^I_+ only transforms under the $\Lambda_{\alpha\beta}$-type gauge transformations while is invariant under the $\Lambda_{+\alpha}$-type transformations.

On the other hand, the Hamiltonian becomes Hermitian only if X^I_+ and X^I_- are equal up to a sign, that is when X^I_+ is real or pure imaginary. Therefore, if we fix the $\Lambda_{\alpha\beta}$-gauge such that

$$X^I_+ = \pm X^I_-,$$

the Hamiltonian becomes Hermitian. To fix the sign choice in the above gauge fixing expression we choose the gauge such that the positivity of the Hamiltonian (the potential) is ensured. It is straightforward to check that this is fulfilled with the negative sign. The appropriate *Hermitian-gauge* fixing condition is then[7]

$$X^I_+ \ + X^I_- = 0 \ . \tag{5.11}$$

One should note that the above gauge fixing condition only partially fixes the $\Lambda_{\alpha\beta}$ gauge symmetry.[8] Besides the Hermiticity problem of the Hamiltonian, the above gauge also removes half of the degrees of freedom in $X^I_\pm$. Hereafter, we will work in the *Hermitian gauge* and define

$$Y^I \equiv -\frac{i}{2}(X^I_+ - X^I_-) = -iX^I_+ \ . \tag{5.12}$$

After fixing the $\Lambda_{\alpha\beta}$-type gauge transformations, we only remain with $\Lambda_{+\alpha}$. For this restricted gauge symmetry the gauge transformations are

$$\begin{aligned} \delta\Phi_\alpha &= 2f^{\beta\gamma}{}_\alpha\Lambda_{+\beta}\Phi_\gamma \\ \delta A_{i\ +\alpha} &= \partial_i\Lambda_{+\alpha} + 2f^{\beta\gamma}{}_\alpha\Lambda_{+\gamma}\ A_{i\ +\beta} \\ \delta A_{i\ \alpha\beta} &= -f^{\rho\gamma}{}_{[\alpha}\Lambda_{\beta]+}\ A_{i\ \rho\gamma}. \end{aligned} \tag{5.13}$$

[7] To fix the gauge condition (5.11) we in fact need at least eight gauge parameters. Therefore, our arguments works for $dim\mathcal{H} \geq 8$. As will become clear in the next subsection the appropriate $\mathcal{H}$ for N M2-branes is $su(N)$, this corresponds to $N \geq 3$. For the special case of $N = 2$, which as discussed in the end of section 4 is equivalent to the $so(4)$-based algebras with an appropriate change of basis, one can explicitly show that in this specific gauge the two Lorenztian and Euclidean descriptions are indeed identical, of course once an $su(2)$ part of the $so(4)$ gauge symmetry of the latter case is also fixed.

[8] Noting the comments in footnote 5, only for $N = 3$ these gauge transformations can be completely fixed by (5.11).

JHEP12(2008)037

After the following renaming

$$\hat{\Lambda}_\alpha = \frac{1}{2}\Lambda_{+\alpha}\,, \quad \hat{A}_{i\ \alpha} = \frac{1}{2}A_{i\ +\alpha}\,, \quad \hat{B}_{i\ \gamma} = f^{\alpha\beta}{}_{\gamma} A_{i\ \alpha\beta}, \tag{5.14}$$

the above gauge transformations take the familiar form of standard gauge transformations for the algebra $\mathcal{H}$ with $\hat{A}_i$ as the gauge field and the two "matter fields" Φ_α and $\hat{B}_{i\ \alpha}$ in the adjoint (and anti-adjoint) representations:

$$\begin{aligned} \delta\Phi &= [\hat{\Lambda}, \Phi]\,, \\ \delta\hat{B}_i &= -[\hat{\Lambda}, \hat{B}_i]\,, \\ \delta\hat{A}_i &= \hat{D}_i\hat{\Lambda} = \partial_i\hat{\Lambda} - [\hat{\Lambda}, A_i] \end{aligned} \tag{5.15}$$

where

$$([\hat{\Lambda}, \Phi])_\alpha = f^{\beta\gamma}{}_{\alpha}\hat{\Lambda}_\beta\Phi_\gamma.$$

It is evident that Y^I are singlets and does not transform under the above gauge transformations of the $\mathcal{H}$ algebra. As we see after fixing the *Hermitian gauge* the proposed $D = 3$, $\mathcal{N} = 8$ action (5.4) written in terms of hatted fields and Y^I (5.12) exhibits a standard $\mathcal{H}$ invariance (with the gauge transformations (5.15)).

5.3 Connection to multi M2-brane theory, the parity invariance

The proposed BL $D = 3$, $\mathcal{N} = 8$ theory is expected to be related to theory of multiple M2-branes in an eleven dimensional flat space background. As such, one then expects that this theory should have the same form for a system of M2-branes and anti-M2 branes. From the worldvolume theory viewpoint M2-branes and anti M2-branes are related by the worldvolume parity and hence the proposed BL theory should be parity invariant [14, 17, 27]. In terms of our four-bracket and the algebra $\mathcal{G}$, the parity invariance is respected if the parity is defined as

$$\sigma^0, \sigma^1 \to \sigma^0, \sigma^1, \quad \sigma^2 \to -\sigma^2, \quad T^\pm \to -T^\pm, \quad T^\alpha \to T^\alpha, \tag{5.16}$$

(σ^0, σ^1 and σ^2 are M2-brane worldvolume coordinates) while X^I behave as scalars under parity, Ψ as a $3d$ fermion, and A_μ as a $3d$ vector. That is, under parity

$$\begin{gathered} X^I_\alpha \to X^I_\alpha\,, \quad X^I_\pm \to -X^I_\pm\,, \\ (A_0, A_1, A_2)_{\ \alpha\beta} \to (A_0, A_1, -A_2)_{\ \alpha\beta}\,, \quad (A_0, A_1, A_2)_{\ +\alpha} \to (-A_0, -A_1, A_2)_{\ +\alpha}\,. \end{gathered} \tag{5.17}$$

As we see the parity (5.16) is an automorphism on the algebra $\mathcal{G}$ as well as its subset $\mathcal{K}$ over which the four-bracket closes (in the relaxed closure sense). More precisely, under the above parity the $\mathcal{H} \in \mathcal{G}$ is invariant, while on the $su(2) \in \mathcal{G}$ it acts as an automorphism.

It is also immediate to check that with (5.17) the action (5.4) is parity invariant. Moreover, the *Hermitian gauge* (5.11) is preserved under parity. This is a necessary condition to have a consistent (Hermitian) multi M2-brane theory.

So far, for the Lorentzian case we have not identified the algebra $\mathcal{H}$ and the underlying algebra $\mathcal{G}$. In the next section we will argue that the choice $\mathcal{H} = su(N)$, $\mathcal{G} = su(2N)$ corresponds to the low energy limit of N M2-branes.

JHEP12(2008)037

5.4 Analysis of 1/2 BPS states

To relate the above "gauge fixed relaxed BLG model" to the theory of multiple M2-branes, we need to specify the algebra $\mathcal{H}$ and relate that to the number of M2-branes N. This can be done by studying the half BPS configurations of the model, the moduli space of which should be identified with the moduli space of N membranes in eleven dimensional flat background, which is $\mathbb{R}^{8N}/S_N$.

The half BPS sector is the one for which the right-hand-side of supersymmetry transformations (5.6) vanishes for any arbitrary supersymmetry transformation parameter ϵ. In order δX^I and δA_{iab} to vanish we need to turn off the fermionic field Ψ. We are then left with the fermionic transformation which has two terms. These terms come with different matrix structure in $so(2,1)$ and $so(8)$ gamma-matrices. Therefore, for $\delta\Psi$ to vanish for any ϵ each term should vanish independently, i.e.

$$D_i X^I = 0\,, \tag{5.18a}$$

$$[X^I, X^J, X^K, T^-]_{\text{phys}} = 0. \tag{5.18b}$$

Recalling the equations of motion (5.5), demanding vanishing of (5.18a), the field strength of the gauge field vanishes and one can always work in a gauge in which $A_i = 0$, and hence (5.5) implies that $\partial_i X^I = 0$ In other words 1/2 BPS membranes must be flat membranes with worldvolume $\mathbb{R}^{2,1}$. We are then left with (5.18b) which recalling the definition of the four-bracket, is satisfied if and only if

$$[X^I, X^J] = 0\ . \tag{5.19}$$

Note that since we are working in the "physical Hermitian gauge" X^I in the above equation have components only along the T^α directions. Therefore, (5.19) is only satisfied when X^I are in Cartan subalgebra of $\mathcal{H}$ and that the number of such possible X^I matrices is $rank(\mathcal{H})-1$. (Note that we have already taken out the "center of mass" degree of freedom in X^I_+.) Noting that X^I are basically related to the position of M2-branes, this means that number of M2-branes N minus one is to be taken as rank of $\mathcal{H}$.

As discussed in [16, 17] (see also [18]) another test for the theory of multi M2-branes is that upon "compactification" it should reproduce theory of multi D2-branes. This together with the above discussions fixes $\mathcal{H} = \tilde{\mathcal{H}} = su(N)$ in its fundamental $N\times N$ representation as the theory of N membranes and hence the underlying algebra $\mathcal{G} = su(2N)$. With this choice it is evident that the moduli space of solutions to (5.19) is the desired R^{8N}/S_N.

Let us discuss some low-lying N's in more detail. The $N=1$ corresponds to a single M2-brane. In this case the fields are 2×2 matrices and therefore all the four-brackets vanish. In this case, as expected, we are dealing with a non-interacting free theory and the only remaining degree of freedom are Y^I (and their fermionic counterparts). This is suggesting that Y^I should correspond to the center of mass degree of freedom in the $N>1$ cases.[9]

[9] It is worth noting that under parity $Y^I \to -Y^I$ and hence the parity transformation we have introduced here besides changing an M2-brane to an anti M2, also acts as a parity on the target space directions transverse to the brane. In the static gauge for the M2-brane, this means that under our parity we are essentially

The next case is $N = 2$ corresponding to two M2-brane system which has X^I_α fields in the adjoint of $su(2)$ plus the Y^I's which are $su(2)$ singlets. Here we are dealing with 4×4 representation of $su(4)$. As discussed this case also makes connection with the $so(4)$-based algebras which have also been discussed to correspond to the two M2-brane dynamics. To argue for the claim that Y^I are the center of mass coordinates one should show that they decouple from the dynamics. The first steps toward this end has been taken in [20, 21], further arguments in support of this is postponed to future works [28]. Once this claim is established for $N = 2$, the same argument can then be generalized to a generic N.

6. Discussion and outlook

Superconformal field theories in three dimensions are clearly of interest, and it is important to understand the necessary requirements underlying their construction. Given the success of the notion of 3-algebras to achieve this, in this paper we have focused in finding finite dimensional matrix representations for these 3-algebras and analysed whether the satisfaction of the so called fundamental identity and closure of the 3-algebra could be relaxed in any way while preserving the main physical features of these theories.

Concerning the first point above, we explored the idea that the non-associative three-algebras and their representations can be expressed in terms of inherently associative classical Lie-algebras (and their matrix representations), by introducing the "non-associative bracket structure" on these algebras; we denoted this underlying associative matrix algebra by $\mathcal{G}$. We argued that to keep the essential properties of the non-associative three-brackets, when expressed in terms of matrices, we need to replace the three-bracket with a four-bracket which is defined as the totally anti-symmetric product of matrices appearing in the bracket. In this procedure, we then need to introduce a given extra matrix, which was called T^-, when moving from a three-bracket to a four-bracket. (T^- is of course an element in $\mathcal{G}$.)

With the working assumption that T^- should anti-commute with all the elements of the "three-algebra", we examined the necessary closure and fundamental identity. As argued, however, one can still have the notion of physically interesting three-algebras if we relax both the fundamental identity and closure conditions in a very particular way. This was done by demanding the closure of the brackets up to the *spurious* parts of the elements of the algebra. In other words, any element has a *physical* as well as a *spurious* part and only bracket of physical parts of the elements lead to a physical element, and the physical part of brackets satisfy the fundamental identity. With this extended, generalized or relaxed notion of fundamental identity and the closure we hence defined the *relaxed-three-algebras* $\mathcal{RA}_3$.

As we showed the above definition of relaxed-three-algebras is still restrictive enough to fix the possible underlying algebra $\mathcal{G}$ and its representations. We showed that within our working assumptions only two cases are possible, one corresponding to the case with positive definite metric on the relaxed-three-algebra, the Euclidean case, and the other

changing sign on nine space coordinates of the eleven dimensional background. This transformation is also a symmetry of the eleven dimensional supergravity and expected to be symmetry of the M2-brane theory too.

with a Lorentzian metric on that algebra. We should emphasize that in our analysis we did not assume anything about the signature of the metric and this condition appeared as the consistency condition within our setting. Moreover, as discussed there is nothing inherently Lorentzian in the underlying algebra $\mathcal{G}$ and the Lorentzian signature is as an artifact of the choice of the set of generators of $\mathcal{G}$ which appear in the four-brackets. This is the resolution to the problem of the negative kinetic energy states (ghosts) in the usual treatment of the Lorentzian BL theory [20 − 22]. For the Euclidean case, using the results of [15], we concluded that there is only one possibility which was called the "$so(4)$-based" algebras for which the underlying algebra is $su(4)$. For the Lorenzian case, however, we showed that there remains a freedom in choosing the algebra which was then fixed once the setting of relaxed-three-algebras was employed in the multi M2-brane theory. As argued the Euclidean case can be formulated without the spurious parts for elements, whereas spurious parts are necessary for the Lorentzian case.

In the corresponding physical model the spurious parts of the fields do not appear at all and the Hilbert space of physical states is hence defined by modding out the total Hilbert space by the spurious parts. As discussed in the specific physical model of multi M2-branes the spurious parts are reminiscent of usually overlooked gauge symmetries. This spurious parts are very similar to the same concept in the context of $2d\ CFT$'s and in string theory [23]. Exploring and understanding these symmetries seems to be an important clue to better understanding of, and resolution to, one of the fundamental open issues in the Bagger-Lambert multi M2-brane theory for more than two M2-branes.

Analyzing the moduli space of 1/2 BPS states of the new realization of the BL-theory in terms of four-brackets, we argued that in order this moduli space to be the same as what is expected from N M2-branes in flat 11 dimensional background, the underlying algebra $\mathcal{G}$ must be taken $su(2N)$ and the physical fields and states must be labeled by physical $N \times N$ representation of $su(N)$.

Our matrix realisation should hopefully not just be thought of another algebraic construction, but as an attempt to achieve a more intuitive physical picture for the effective field theory governing coincident M2-branes. In this respect, we would like to clarify the role played by our $su(2N)$ underlying algebra and the $su(N)$ $\mathcal{H}$ algebra in an analogous way to what we have for multiple (coincident) D-branes, where the degrees of freedom corresponding to open string attached to and stretched between parallel D-branes leads to the $su(N)$ structure [29]. Note that to get the $su(N)$ structure we should remember that open strings stretched between D-branes come in two opposite orientations each of which includes a massless (vector) state when two D-branes become coincident. For the case of M2-branes, similar to the D-brane case, we have open M2-branes stretched between two M2-branes. Although we do not know the spectrum of open M2-branes as well as we want to, it is expected that there are massless states in the coincident M2-brane limit. Again similarly to the stretched open string case, there are open M2 and anti-M2 branes. Recalling that M2-branes are two dimensional (to be compared with one dimensional strings), for the case of membranes there are two options to get an anti-M2 brane for a given M2; the M2-brane and anti-M2-brane are related by parity on the worldvolume of the brane. This is suggestive that when we consider the four possible open M2 and anti-M2 branes

JHEP12(2008)037

(two M2-brane in which orientation on both directions have changed with respect to each other and two respective anti-M2-branes which are related by worldvolume parity to the two M2-brane cases) we are over-counting the degrees of freedom and this should be mod out by the worldvolume parity. In other words, the reduction from $su(2N)$ to $su(N)$ which labels N M2-brane fluctuations could be done through worldvolume parity. As argued the parity on the M2-brane worldvolume is acting as an automorphism of this $su(2N)$ algebra and keeps the $su(N)$ labels of the physical states/fields invariant. It would be very interesting to make the above picture more precise and concrete [28].

As discussed the ghost problem of the Lorentzian three-algebras in our setup manifested itself in our setup as non-hermiticty of the Hamiltonian before the gauge fixing and can be removed once we fix the *Hermitian gauge*. This resolution which is in accord with proposal in [20, 21], however, requires identifying the mode, which we called Y^I as the center of mass degree of freedom of M2-brane system. The problem which is still remaining in this direction is establishing the fact that the center of mass degree of freedom is indeed decoupled. Once this problem is settled, our setup which is based on usual matrices provides the needed tools to make further analysis of the $D = 3$, $\mathcal{N} = 8$ or the multi-M2-brane theory.

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JHEP12(2008)037

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